Exercise 19

Solve the differential equation using (a) undetermined coefficients and (b) variation of parameters.

$$4y'' + y = \cos x$$

Solution

Since the ODE is linear, the general solution can be written as the sum of a complementary solution and a particular solution.

$$y = y_c + y_p$$

The complementary solution satisfies the associated homogeneous equation.

$$4y_c'' + y_c = 0 \tag{1}$$

This is a linear homogeneous ODE, so its solutions are of the form $y_c = e^{rx}$.

$$y_c = e^{rx} \quad \rightarrow \quad y_c' = re^{rx} \quad \rightarrow \quad y_c'' = r^2 e^{rx}$$

Plug these formulas into equation (1).

$$4(r^2 e^{rx}) + e^{rx} = 0$$

Divide both sides by e^{rx}.

$$4r^2 + 1 = 0$$

Solve for r.

$$r = \left\{-\frac{i}{2}, \frac{i}{2}\right\}$$

Two solutions to the ODE are $e^{-ix/2}$ and $e^{ix/2}$. By the principle of superposition, then,

$$\begin{aligned} y_c(x) &= C_1 e^{-ix/2} + C_2 e^{ix/2} \\ &= C_1 \left(\cos\frac{x}{2} - i\sin\frac{x}{2}\right) + C_2\left(\cos\frac{x}{2} + i\sin\frac{x}{2}\right) \\ &= (C_1 + C_2)\cos\frac{x}{2} + (-iC_1 + iC_2)\sin\frac{x}{2} \\ &= C_3\cos\frac{x}{2} + C_4\sin\frac{x}{2}. \end{aligned}$$

On the other hand, the particular solution satisfies the original ODE.

$$4y_p'' + y_p = \cos x \tag{2}$$

Part (a)

Since the inhomogeneous term is a cosine, the particular solution is

$$y_p = A\cos x + B\sin x.$$

Note that because there are only even derivatives on the left side, $B = 0$.

$$y_p = A\cos x \quad \rightarrow \quad y_p' = -A\sin x \quad \rightarrow \quad y_p'' = -A\cos x$$

Substitute these formulas into equation (2).

$$4(-A\cos x) + (A\cos x) = \cos x$$

$$(-4A + A)\cos x = \cos x$$

Match the coefficients on both sides to get an equation for A.

$$-4A + A = 1$$

Solving it yields

$$A = -\frac{1}{3},$$

which means the particular solution is

$$y_p = -\frac{1}{3}\cos x.$$

Therefore, the general solution to the ODE is

$$\begin{aligned} y(x) &= y_c + y_p \\ &= C_3\cos\frac{x}{2} + C_4\sin\frac{x}{2} - \frac{1}{3}\cos x, \end{aligned}$$

where C_3 and C_4 are arbitrary constants.

Part (b)

In order to obtain a particular solution, use the method of variation of parameters: Allow the parameters in the complementary solution to vary.

$$y_p = C_3(x)\cos\frac{x}{2} + C_4(x)\sin\frac{x}{2}$$

Differentiate it with respect to x.

$$y_p' = C_3'(x)\cos\frac{x}{2} + C_4'(x)\sin\frac{x}{2} - \frac{1}{2}C_3(x)\sin\frac{x}{2} + \frac{1}{2}C_4(x)\cos\frac{x}{2}$$

If we set

$$C_3'(x)\cos\frac{x}{2} + C_4'(x)\sin\frac{x}{2} = 0, \tag{3}$$

then

$$y_p' = -\frac{1}{2}C_3(x)\sin\frac{x}{2} + \frac{1}{2}C_4(x)\cos\frac{x}{2}.$$

Differentiate it with respect to x once more.

$$y_p'' = -\frac{1}{2}C_3'(x)\sin\frac{x}{2} + \frac{1}{2}C_4'(x)\cos\frac{x}{2} - \frac{1}{4}C_3(x)\cos\frac{x}{2} - \frac{1}{4}C_4(x)\sin\frac{x}{2}$$

Substitute these formulas into equation (2).

$$4\left[-\frac{1}{2}C_3'(x)\sin\frac{x}{2} + \frac{1}{2}C_4'(x)\cos\frac{x}{2} - \cancel{\frac{1}{4}C_3(x)\cos\frac{x}{2}} - \cancel{\frac{1}{4}C_4(x)\sin\frac{x}{2}}\right] + \left[\cancel{C_3(x)\cos\frac{x}{2}} + \cancel{C_4(x)\sin\frac{x}{2}}\right] = \cos x$$

Simplify the result.

$$-C_3'(x)\sin\frac{x}{2} + C_4'(x)\cos\frac{x}{2} = \frac{1}{2}\cos x \tag{4}$$

Multiply both sides of equation (3) by $\sin(x/2)$, and multiply both sides of equation (4) by $\cos(x/2)$.

$$C_3'(x)\cos\frac{x}{2}\sin\frac{x}{2} + C_4'(x)\sin^2\frac{x}{2} = 0$$

$$-C_3'(x)\cos\frac{x}{2}\sin\frac{x}{2} + C_4'(x)\cos^2\frac{x}{2} = \frac{1}{2}\cos x\cos\frac{x}{2}$$

Add the respective sides of these equations to eliminate $C_3'(x)$.

$$C_4'(x) = \frac{1}{2}\cos x\cos\frac{x}{2} \tag{5}$$

Integrate this result to get $C_4(x)$, setting the integration constant to zero.

$$\begin{aligned}
C_4(x) &= \int^x C_4'(w)\,dw \\
&= \int^x \frac{1}{2}\cos w\cos\frac{w}{2}\,dw \\
&= \int^{x/2} \frac{1}{2}\cos 2u\cos u\,(2\,du) \\
&= \int^{x/2} \cos 2u\cos u\,du \\
&= \int^{x/2} (1 - 2\sin^2 u)\cos u\,du \\
&= \int^{\sin(x/2)} (1 - 2v^2)\,dv \\
&= \left(v - \frac{2}{3}v^3\right)\Big|^{\sin(x/2)} \\
&= \sin\frac{x}{2} - \frac{2}{3}\sin^3\frac{x}{2}
\end{aligned}$$

Multiply both sides of equation (3) by $\cos(x/2)$, and multiply both sides of equation (4) by $\sin(x/2)$.

$$C_3'(x)\cos^2\frac{x}{2} + C_4'(x)\cos\frac{x}{2}\sin\frac{x}{2} = 0$$

$$-C_3'(x)\sin^2\frac{x}{2} + C_4'(x)\cos\frac{x}{2}\sin\frac{x}{2} = \frac{1}{2}\cos x\sin\frac{x}{2}$$

Subtract the respective sides of these equations to eliminate $C_4'(x)$.

$$C_3'(x) = -\frac{1}{2}\cos x\sin\frac{x}{2} \tag{6}$$

Integrate this result to get $C_3(x)$, setting the integration constant to zero.

$$\begin{aligned}
C_3(x) &= \int^x C_3'(w)\,dw \\
&= -\int^x \frac{1}{2}\cos w \sin\frac{w}{2}\,dw \\
&= -\int^{x/2} \frac{1}{2}\cos 2u \sin u\,(2\,du) \\
&= -\int^{x/2} \cos 2u \sin u\,du \\
&= -\int^{x/2} (2\cos^2 u - 1)\sin u\,du \\
&= -\int^{\cos(x/2)} (2v^2 - 1)(-dv) \\
&= \left(\frac{2}{3}v^3 - v\right)\Bigg|^{\cos(x/2)} \\
&= \frac{2}{3}\cos^3\frac{x}{2} - \cos\frac{x}{2}
\end{aligned}$$

Therefore,

$$\begin{aligned}
y_p &= C_3(x)\cos\frac{x}{2} + C_4(x)\sin\frac{x}{2} \\
&= \left(\frac{2}{3}\cos^3\frac{x}{2} - \cos\frac{x}{2}\right)\cos\frac{x}{2} + \left(\sin\frac{x}{2} - \frac{2}{3}\sin^3\frac{x}{2}\right)\sin\frac{x}{2} \\
&= \frac{2}{3}\cos^4\frac{x}{2} - \cos^2\frac{x}{2} + \sin^2\frac{x}{2} - \frac{2}{3}\sin^4\frac{x}{2} \\
&= \frac{2}{3}\left(\cos^4\frac{x}{2} - \sin^4\frac{x}{2}\right) - \left(\cos^2\frac{x}{2} - \sin^2\frac{x}{2}\right) \\
&= \frac{2}{3}\left(\cos^2\frac{x}{2} + \sin^2\frac{x}{2}\right)\left(\cos^2\frac{x}{2} - \sin^2\frac{x}{2}\right) - \left(\cos^2\frac{x}{2} - \sin^2\frac{x}{2}\right) \\
&= \frac{2}{3}(1)(\cos x) - (\cos x) \\
&= -\frac{1}{3}\cos x,
\end{aligned}$$

and the general solution to the ODE is

$$\begin{aligned}
y(x) &= y_c + y_p \\
&= C_3\cos\frac{x}{2} + C_4\sin\frac{x}{2} - \frac{1}{3}\cos x,
\end{aligned}$$

where C_3 and C_4 are arbitrary constants.